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EXTRACT OF A LETTER TO MR. SYLVESTER FROM PROF. CLIFFORD OF UNIVERSITY COLLEGE, LONDON.

THE subjoined matter is so exceedingly interesting and throws such a flood of light on the chemico-algebraical theory, that I have been unable to resist the temptation to insert it in the *Journal*, without waiting to obtain the writer's permission to do so, for which there is not time available between the date of its receipt and my proximate departure for Europe. It is written from Gibraltar, whither Professor Clifford has been ordered to recruit his health, a treasure which he ought to feel bound to guard as a sacred trust for the benefit of the whole mathematical world. J. J. S.

"The new Journal I look forward to with the greatest interest: it will be the only English periodical in which one will have room to print formulæ, except the Philosophical Transactions. I had designed for you a series of papers on the application of Grassmann's methods, but there is only one of them fit for printing yet. It is an *explanation* of the laws of quaternions and of my biquaternions by resolving the units into factors having simpler laws of multiplication; a determination of the corresponding systems for space of any number of dimensions; and a proof that the resulting algebra is a *compound* (in Peirce's sense) of quaternion algebras. It thus appears that quaternions are the last word of geometry in regard to complex algebras.

"Another of them was to be about the very thing you speak of, which was communicated to the British Association at Bristol, *not* Bradford. There is no question of reclamation, because the whole thing is really no more than a translation into other language of your own theories published ages ago in the Cambridge Mathematical Journal.* I have a strong impression that you will find there the analogy of covariants and invariants to compound radicals and saturated molecules.*

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*Not having access to the reports of the British Association, I wrote to Professor Clifford to the effect that I had an impression that he had made a communication to the meeting at Bradford which had some affinity to my chemico-algebraical theory and requesting him, in that case, to claim back what was his own.

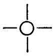
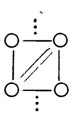
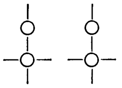
I think he overstates the obligations which he alleges to my previous papers. At all events he has more than reconquered his title to the merit of the first conception by the completeness he has imparted to it, for whilst I was only able, in certain cases, to represent in terms of the roots of the parent quantic, the quantitative constitution of a form pictured by a graph, through the instrumentality of the reciprocal figure, he has given a direct rule for finding in all cases the algebraical content of a graph in terms of the coefficients, from the graph itself. In a word he has found the universal pass key to the *quantification of graphs.* It seems to me in the highest degree improbable that our joint speculations should not eventually find their embodiment in chemical doctrine proper, and I think that young chemists desirous to raise their science to its proper rank would act wisely in making themselves masters betimes, of the theory of algebraical forms. What mechanics is to physics, that I think algebraical morphology, founded at option on the theory of partitions or ideal elements, or both, is destined to be to the chemistry of the future. Brodie, in his Chemical Calculus, seems to have had an instinctive appreciation of this truth. I have previously called attention to the fact that invariants and isomerism are sister theories.

"I consider forms which are linear in a certain number of sets of k variables each. To fix the ideas, suppose $k = 2$ and that I have altogether 6 sets of 2 variables each, namely

$$x_1x_2\ ,\quad y_1y_2\ ,\quad z_1z_2\ ,\quad u_1u_2\ ,\quad v_1v_2\ ,\quad w_1w_2\ .$$

Suppose the forms are

$$(xyzu)\ ,\quad (yzvw)\ ,\quad (xv)\ ,\quad (uw)\ ;$$

viz. $(xyzu)$ means an expression separately linear and homogeneous in the x, the y, the z, and the u, and so for the rest. I observe that in these four forms each set of variables occurs twice. This being so, there is one invariant of the four forms, which is invariant in regard to *independent* transformations of the six sets of variables. This you knew thirty years ago. All I add is:† *to obtain this invariant*, regard the *variables as alternate numbers*, and *simply multiply all the forms together*. By *alternate* numbers‡ I mean those whose multiplication is polar ($xy = -yx$) and whose squares are zero. The product of the forms will then be equal to the invariant in question multiplied by the product of all the variables. The quartic forms may be represented by the symbol , the quadratics by -o- . Thus the invariant $(xyzu)(yzvw)(xv)(uw)$ will be represented by the figure ; whereas, $(xyzu)$ $(yzvw)(xu)(vw)$ is this form o=o=o=o . The former is clearly the product of the two quartic covariants  got by cutting it across the dotted lines; while the latter is the product of the quadricovariants o=o= , =o=o . A *bond* between two forms means a set of variables common to

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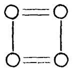
[*] I feel induced to say of Professor Clifford what a great master in another field is reported to have done of a certain famous "Oxford graduate," "That young man sees many things in my works which I was not before aware was contained in them," and to regard his generous confessions of obligation as the graceful homage of a young athlete to a half-spent veteran in the arena. What I have seen as in a glass darkly by the uncertain light of analogy, he has viewed by direct vision and brought out into the full blaze of day.

[†] "All that Professor Clifford adds" is the very pith and marrow of the matter which before was wanting.

[‡] The term *alternate numbers* (Hankel's, if my memory serves me right) applied to letters, subject (like Hamilton's i, j, k,) to the law of polar multiplication, seems to me very inadvisable to introduce into the subject. There is no question of quantity, still less of number; and the epithet *alternate* has no meaning except by oblique reference to alternants in the sense of determinants. I think the right term is that first employed by myself and which I shall certainly continue to employ undeviatingly, viz. "polar elements." In like manner I protest against my most expressive and suggestive word "cumulants" being ignored by Mr. Muir and replaced by the unmeaning and ill chosen word "continuants." J. J. S.

them. Of course, we may regard two or more of the forms as identical, and so form invariants of a single form; thus  is the discriminant of a cubic.* . . . . . . Of course, the main thing is to pass from this system of separate variables to that in which the same variables occur to higher orders in the same form, or back again — what you call 'unravelment.' . . . . . .

"The part of the theory which astonished me most is its application to *intergredient* variables when the number in a set is greater than 3, — such as the six co-ordinates of a line in the case of quaternary forms. When the original variables are regarded as alternate numbers, these intergredients are simply their binary products. Thus by simply multiplying the linear forms representing two planes, we get an intergredient form representing their line of intersection. And so generally, whatever be the number of variables in a set, the intergredient variables are merely their products so many together. With this understanding, the product of a set of forms in which the variables are regarded as alternate numbers is the *only* invariant or covariant of the forms which possess certain definite characters of invariance.

"The ordinary theory of symmetrical forms seems to me to bear the same relation to this one (of forms linear in several sets of variables) that a boulder does to a crystal — all the angles rounded off so that you can't see through it so clearly. . . . . . "

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* I will take the example of this figure to illustrate Professor Clifford's rule for finding the *algebraical content* of a graph. Let the bonds be called $\begin{smallmatrix}x\\y\end{smallmatrix}\ z\ \begin{smallmatrix}t\\u\end{smallmatrix}\ v$. Then there will be 4 forms corresponding to the 4 apices or atoms, viz.

$$\begin{array}{l}(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)(x_1, x_2)(y_1, y_2)(z_1, z_2),\\(b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8)(z_1, z_2)(t_1, t_2)(u_1, u_2),\\(c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8)(t_1, t_2)(u_1, u_2)(v_1, v_2),\\(d_1, d_2, d_3, d_4, d_5, d_6, d_7, d_8)(v_1, v_2)(x_1, x_2)(y_1, y_2),\end{array}$$

where all the x, y, z, t, u, v letters are to be regarded as *polar elements*. Take the polar product of these forms; the coefficient of

$$x_1 . x_2 . y_1 . y_2 . z_1 . z_2 . t_1 . t_2 . u_1 . u_2 . v_1 . v_2$$

will be an invariant of 3 lineo-lineo-linear forms.

If we make the values identical for the same index, whatever the letter which it affects, it becomes an invariant of a single lineo-lineo-linear form; and finally, if we make the coefficients of $x_1y_1z_2$, $y_1z_1x_2$, $z_1x_1y_2$ all alike, and again the coefficients of $x_2y_2z_1$, $y_2z_2x_1$, $z_2x_2y_1$ all alike, and identify the letters x, y, z, the form becomes a binary cubic and the invariant becomes its discriminant. [*Quaere*, whether this beautiful use of the method of polar multiplication is not, in its ultimate essence, identical with Professor Cayley's original method of hyperdeterminants.] We know *à priori* by my permutation-sum test that the algebraical content above indicated will not vanish because $\Sigma\,(a-b)^2\,(a-d)^2\,(a-c)\,(b-d)$ is not zero, whereas the algebraical content of the figure formed by turning round one of each pair of the doubled lines into the position of the two diagonals respectively *will* vanish because the permutation-sum of $\Sigma\,(a-b)\,(b-c)\,(c-d)\,(d-a)\,(a-c)\,(b-d)$ is zero.

J. J. S.